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Quantum Mechanics in Finite Dimensions

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We explicitly compute, following the method of Weyl, the commutator [Q, P] *of the position operator* Q *and the momentum operator* P *of a particle when the dimension of the space on which they act is finite with a discrete spectrum; and we show that in the limit of a continuous spectrum with the dimension going to infinity this reduces to the usual relation of Heisenberg.*

1. INTRODUCTION

Weyl[1] has shown that the Schrödinger representation is a necessary consequence of Heisenberg's commutation relation. He proves this using the ray representations of the Abelian group of rotations and an ingenious limiting process to go from finite rotations in ray space to a two-parameter continuous group. This approach has been particularly emphasized by Schwinger,[2] who has also shown that such operators form a complete set and furnish a basis for measurement symbols. More recently, Ramakrishnan[3] and his collaborators have studied exhaustively the representation theory of generalized Clifford algebra, which yields the ray representations of the Abelian group of rotations.

In this paper we derive, by limiting to the case of finite dimensions, the commutator $[Q, P]$, where Q and P are the position and momentum operators respectively. We show that by going to the limit of continuous parametrization (valid as the dimension goes to infinity), we recover the standard Heisenberg relations.

We believe that this work will open the possibility of studying quantum mechanics in finite-dimensional space with a discrete spectrum.

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2. WEYL'S FORM OF THE HEISENBERG RELATION

Suppose A and B are two elements of the Abelian group of unitary rotations in a ray space so that

$$AB = \omega BA \tag{1}$$

where ω is the primitive nth root of unity. By iteration we have

$$A^k B^l = \omega^{kl} B^l A^k \tag{2}$$

from which it follows that A^n commutes with B and B^n commutes with A; and if the representation is irreducible, it follows from Schur's lemma that

$$A^n = I, \qquad B^n = I \tag{3}$$

In the diagonal representation for B, i.e.,

$$B = \mathrm{diag}(1, \omega, \omega^2, ..., \omega^{n-1}) \tag{4}$$

A has the form[1] of a cyclic permutation matrix

$$A = \begin{pmatrix} 0100 \cdots 0 \\ 0010 \cdots 0 \\ \vdots \\ 0000 \cdots 1 \\ 1000 \cdots 0 \end{pmatrix} \tag{5}$$

The interesting properties of the algebra satisfied by operators like A and B, which is a generalization of the usual Clifford algebra, have been studied by Schwinger,[2] Morinaga and Nono,[4] Yamazaki,[5] and Morris,[6] while its connection with physical problems through the study of specific representations has been systematically carried out by Ramakrishnan and collaborators.[2]

If one identifies

$$A = e^{i\xi P}, \qquad B = e^{i\eta Q} \tag{6}$$

where ξ and η are arbitrary real parameters, then it follows that Eq. (6) is the Weyl form of the Heisenberg commutation relation

$$[Q, P] = iI \tag{7}$$

if we allow power series expansion of operator exponentials (which is justified if A and B are bounded, but not otherwise; see, e.g., Ref. 7). Weyl takes the limit of infinitesimal ζ and η with $n \to \infty$ such that $\eta\xi\eta = 2\pi$ to show that

$$P = (1/i)\, \partial/\partial q \tag{8}$$

3. FINITE DIMENSIONS

We now solve Eq. (6) for P and Q by taking logarithms. That is, given Eq. (6) where A and B satisfy Eqs. (2) and (3), we evaluate the commutator $[Q, P]$ and show that it becomes Eq. (7) in the continuous limit. We take the diagonal form for B given by Eq. (4) and Eq. (5) for A.

Since any circulant matrix like A is diagonalized by the Sylvester matrix S, we have

$$S^{-1}AS = B \tag{9}$$

with

$$S = n^{-1/2} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{n-2} \\ \vdots & & & & \\ 1 & \omega^{n-1} & \omega^{n-2} & \cdots & \omega \end{pmatrix} \tag{9'}$$

and

$$S^{-1} = S^{+} \tag{10}$$

One obtains from Eq. (6), on taking logarithms,

$$i\xi P = \log A \equiv S(\log B)S^{-1} \tag{11}$$

$$i\eta Q = \log B \tag{12}$$

where

$$\log B = (\log \omega)\,\mathrm{diag}(0, 1, 2,..., n-1) \tag{13}$$

Since A and B are nonsingular and diagonalizable, it follows[8] that $\log A$ and $\log B$ exist. Of course, they are multivalued. Elementwise, labeling the rows and columns from 0 to $n - 1$, one gets

$$\begin{aligned} B_{rs} &= \omega^r \delta_{rs}, \qquad & S_{rs} &= n^{-1/2}\omega^{rs} \\ (S^{-1})_{rs} &= n^{-1/2}\omega^{-rs}, \qquad & (\log B)_{rs} &= (\log \omega)\, r\delta_{rs} \end{aligned} \tag{14}$$

Next we consider the commutator

$$K = [i\eta Q, i\xi P] = [\log B, S(\log B)S^{-1}] \tag{15}$$

We have

$$K_{rs} = \frac{(\log \omega)^2}{n}(r - s)\sum_{u=0}^{n-1} u\omega^{u(r-s)} \tag{16}$$

If $\omega^{r-s} = x = 1$, then

$$K_{rs} = \frac{(\log \omega)^2}{n}(r - s)\frac{n(n-1)}{2} \tag{17}$$

If $x \neq 1$, then, since $x^n = 1$, there results

$$\sum_{u=0}^{n-1} ux^u = \frac{n}{x-1} \tag{18}$$

and hence

$$K_{rs} = \frac{(\log \omega)^2}{n} (r - s) \frac{n}{\omega^{r-s} - 1} \tag{19}$$

Thus, we find

$$[Q, P]_{rs} = \frac{(s-r)(\log \omega)^2}{n\xi\eta} \frac{n(n-1)}{2}, \qquad \text{when} \quad \omega^{r-s} = 1 \tag{20}$$

and

$$[Q, P]_{rs} = \frac{(s-r)(\log \omega)^2}{n\xi\eta} \frac{n}{\omega^{r-s} - 1}, \qquad \text{when} \quad \omega^{r-s} \neq 1 \tag{21}$$

We notice that since n is finite, one could choose $\xi = \eta = 1$ and that $[Q, P]$ is off-diagonal and hence trace-free, as it should be for bounded operators.

It can now be proved that the commutation relation given by Eqs. (21) and (22) does indeed yield the Heisenberg commutation relation in the limit as $n \to \infty$. Beginning with Eq. (16), we relabel the rows and columns from $-(n-1)/2$ to $(n-1)/2$ and replace the sum by an integral, that is, we let the matrix index take continuous values.[9,10] Thus the sum

$$[Q, P]_{rs} = -\frac{(\log \omega)^2}{n\xi\eta} \sum_{u=0}^{n-1} u(r-s)\omega^{r-s} \tag{22}$$

reduces in the limit $n \to \infty$ to the integral

$$\begin{aligned} [Q, P]_{rs} &= -\frac{(\log \omega)^2}{n\xi\eta} (r-s) \int_{-\infty}^{\infty} ue^{2\pi iu(r-s)/n}\, du \\ &= -i(r-s) \frac{d}{d(r-s)} \int_{-\infty}^{\infty} e^{2\pi iu(r-s)/n} d\left(\frac{u}{n}\right) \\ &= -i(r-s)\, \delta'(r-s) \\ &= i\delta(r-s) \end{aligned} \tag{23}$$

where we have used $n\xi\eta = 2\pi$ valid when ξ and η are infinitesimal and $n \to \infty$. This completes the proof. The eigenvalues of the operator Q are given by

$$q = k\xi \bmod n\xi \tag{24}$$

when k is any integer. However, since in this limit we have $n\xi\eta = 2\pi$, it follows that $n\xi = 2\pi/\eta$.

As has been pointed out by Weyl, by choosing η infinitesimal, $n\xi$ can be made to approach infinity, so that $q = k\xi$. We use this trick in retaining only the principal value of ($\log \omega$) in this limit.

4. CONCLUSIONS

We have evaluated the commutator $[Q, P]$ when the space on which the operators act is finite and has a discrete spectrum. The commutator for finite n is elevated to what we call "quantum mechanics in finite-dimensional discrete space." It turns out that the commutator is off-diagonal and hence trace-free (as it should be) for finite n. This implies no "uncertainty" and no "zero-point energy" if these concepts have any meaning for finite n. Of course, in the limiting case as n approaches infinity continuously, the commutator becomes strictly diagonal and reduces to a multiple of the Dirac delta function, thus restoring the Heisenberg commutation relations.

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